

Convolution type form of the Ito representation of the infinitesimal generator for Levy processes

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Abstract

In the present paper we show that the Ito representation of the infinitesimal generator L for Levy processes can be written in a convolution type form. Using the obtained convolution form and the theory of integral equations with difference kernels we study the properties of Levy processes.

1 Main notions

Let us introduce the notion of the Levy processes.

Definition 1.1 *A stochastic process $\{X_t : t \geq 0\}$ is called Levy process, if the following conditions are fulfilled:*

1. *Almost surely $X_0 = 0$, i.e. $P(X_0 = 0) = 1$.
(One says that an event happens almost surely (a.s.) if it happens with probability one.)*
2. *For any $0 \leq t_1 < t_2 \dots < t_n < \infty$ the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$*

are independent (independent increments).

(To call the increments of the process X_t independent means that increments $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \dots, X_{t_n} - X_{t_{n-1}}$ are mutually (not just pairwise) independent.)

3. For any $s < t$ the distributions of $X_t - X_s$ and X_{t-s} are equal (stationary increments).

4. Process X_t is almost surely right continuous with left limits.

Then Levy-Khinchine formula gives (see [1], [5])

$$\mu(z, t) = E\{\exp[izX_t]\} = \exp[-t\lambda(z)], \quad t \geq 0, \quad (1.1)$$

where

$$\lambda(z) = \frac{1}{2}Az^2 - i\gamma z - \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|<1})\mu(dx). \quad (1.2)$$

Here $A \geq 0$, $\gamma = \overline{\gamma}$, $z = \overline{z}$ and $\mu(dx)$ is a measure on the axis $(-\infty, \infty)$ satisfying the conditions

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} \mu(dx) < \infty. \quad (1.3)$$

The Levy-Khinchine formula is determined by the Levy-Khinchine triplet $(A, \gamma, \mu(dx))$.

By $P_t(x_0, \Delta)$ we denote the probability $P(X_t \in \Delta)$ when $P(X_0 = x_0) = 1$ and $\Delta \in R$. The transition operator P_t is defined by the formula

$$P_t f(x) = \int_{-\infty}^{\infty} P_t(x, dy) f(y). \quad (1.4)$$

Let C_0 be the Banach space of continuous functions $f(x)$, satisfying the condition $\lim_{|x| \rightarrow \infty} f(x) = 0$, with the norm $\|f\| = \sup_x |f(x)|$. We denote by C_0^n the set of $f(x) \in C_0$ such that $f^{(k)}(x) \in C_0$, $(1 \leq k \leq n)$. It is known that [5]

$$P_t f \in C_0, \quad (1.5)$$

if $f(x) \in C_0^2$.

Now we formulate the following important result (see [5]).

Theorem 1.2 (Levy-Ito decomposition.) *The family of the operators P_t ($t \geq 0$) defined by the Levy process X_t is a strongly continuous semigroup on C_0 with the norm $\|P_t\| = 1$. Let L be its infinitesimal generator. Then*

$$Lf = \frac{1}{2}A \frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + \int_{-\infty}^{\infty} (f(x+y) - f(x) - y \frac{df}{dx} 1_{|y|<1}) \mu(dy), \quad (1.6)$$

where $f \in C_0^2$.

2 Convolution type form of infinitesimal generator

1. In this section we prove that the infinitesimal generator L can be represented in the special convolution type form

$$Lf = \frac{d}{dx} S \frac{d}{dx} f, \quad (2.1)$$

where the operator S is defined by the relation

$$Sf = \frac{1}{2}Af + \int_{-\infty}^{\infty} k(y-x)f(y)dy. \quad (2.2)$$

We note that for arbitrary a ($0 < a < \infty$) the inequality

$$\int_{-a}^a |k(t)|dt < \infty \quad (2.3)$$

is true.

Formula (2.2) was proved before in our works [4] under some additional conditions. In the present paper we omit these additional conditions and prove the formula (2.2) for the general case. The representation of L in form (2.1) is convenient as the operator L is expressed with the help of the classic differential and convolution operators. Using the obtained convolution form of the generator L and the theory of integral equations with difference kernels [3] we investigate the properties of a wide class of Levy processes.

By $C(a)$ we denote the set of functions $f(x) \in C_0$ which have the following property:

$$f(x) = 0, \quad x \notin [-a, a] \quad (2.4)$$

i.e. the function $f(x)$ is equal to zero in the neighborhood of $x = \infty$. We note, that parameter a can be different for different f .

We introduce the functions

$$\mu_-(x) = \int_{-\infty}^x \mu(dx), \quad x < 0, \quad (2.5)$$

$$\mu_+(x) = - \int_x^\infty \mu(dx), \quad x > 0, \quad (2.6)$$

where the functions $\mu_-(x)$ and $\mu_+(x)$ are monotonically increasing on the half-axis $(-\infty, 0]$ and $[0, \infty)$ respectively and

$$\mu_+(x) \rightarrow 0, \quad x \rightarrow +\infty; \quad \mu_-(x) \rightarrow 0, \quad x \rightarrow -\infty. \quad (2.7)$$

We note that

$$\mu_-(x) \geq 0, \quad x < 0; \quad \mu_+(x) \leq 0, \quad x > 0. \quad (2.8)$$

In view of (1.3) the integrals in the right sides of (2.6) and (2.5) are convergent. Hence we have

$$\int_{-\infty}^\infty f(x) \mu(dx) = \int_{-\infty}^0 f(x) d\mu_-(x) + \int_0^\infty f(x) d\mu_+(x). \quad (2.9)$$

Theorem 2.1 *The following relations*

$$\varepsilon^2 \mu_\pm(\pm\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow +0, \quad (2.10)$$

$$- \int_{-a}^0 x \mu_-(x) dx < \infty, \quad - \int_0^a x \mu_+(x) dx < \infty, \quad 0 < a < \infty \quad (2.11)$$

are true

Proof. According to (1.3) we have

$$0 \leq \int_{-a}^{-\varepsilon} x^2 d\mu_-(x) \leq M, \quad (2.12)$$

where M does not depend from ε . Integrating by parts the integral of (2.12) we obtain:

$$\int_{-a}^{-\varepsilon} x^2 d\mu_-(x) = \varepsilon^2 \mu_-(-\varepsilon) - a^2 \mu_-(-a) - 2 \int_{-a}^{-\varepsilon} x \mu_-(x) dx \leq M, \quad (2.13)$$

The function $-\int_{-a}^{-\varepsilon} x \mu_-(x) dx$ of ε is monotonic increasing. According to (2.13) this function is bounded. Hence we have

$$\lim_{\varepsilon \rightarrow +0} \int_{-a}^{-\varepsilon} x \mu_-(x) dx = \int_{-a}^0 x \mu_-(x) dx \quad (2.14)$$

It follows from (2.13) and (2.14) that

$$\lim_{\varepsilon \rightarrow +0} \varepsilon^2 \mu_-(-\varepsilon) = 0. \quad (2.15)$$

Thus, relations (2.10) and (2.11) are proved for $\mu_-(x)$. In the same way relations (2.10) and (2.11) can be proved for $\mu_+(x)$. \square

2. Let us introduce the functions

$$k_-(x) = \int_{-1}^x \mu_-(t) dt, \quad -\infty \leq x < 0, \quad (2.16)$$

$$k_+(x) = - \int_x^1 \mu_+(t) dt, \quad 0 < x \leq +\infty. \quad (2.17)$$

In view of (2.11) the integrals in the right sides of (2.16) and (2.17) are absolutely convergent. From (2.16) and (2.17) we obtain the assertions:

Theorem 2.2 *1. The function $k_-(x)$ is monotonically increasing on the $(-\infty, 0)$ and*

$$k_-(x) \geq 0, \quad -1 \leq x < 0. \quad (2.18)$$

2. The function $k_+(x)$ is monotonically decreasing on the $(0, +\infty)$ and

$$k_+(x) \geq 0, \quad 0 < x \leq 1. \quad (2.19)$$

Further we need the following result:

Theorem 2.3 *The following relations*

$$\varepsilon k_-(-\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow +0; \quad \varepsilon k_+(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow +0; \quad (2.20)$$

$$\int_{-1}^0 k_-(x) dx < \infty; \quad \int_0^1 k_+(x) dx < \infty \quad (2.21)$$

are valid.

Proof. According to (2.11) we have

$$0 \leq - \int_{-1}^{-\varepsilon} x \mu_-(x) dx \leq M, \quad (2.22)$$

where M does not depend from ε . Integrating by parts the integral of (2.22) we obtain:

$$-\int_{-1}^{-\varepsilon} x\mu_{-}(x)dx = \varepsilon k_{-}(-\varepsilon) - k_{-}(-1) + \int_{-1}^{-\varepsilon} k_{-}(x)dx \leq M, \quad (2.23)$$

The function $\int_{-1}^{-\varepsilon} k_{-}(x)dx$ of ε is monotonic increasing. This function is bounded (see (2.23)). Hence we have

$$\lim_{\varepsilon \rightarrow +0} \int_{-1}^{-\varepsilon} k_{-}(x)dx = \int_{-1}^0 k_{-}(x)dx \quad (2.24)$$

It follows from (2.23) and (2.24) that

$$\lim_{\varepsilon \rightarrow +0} \varepsilon k_{-}(-\varepsilon) = 0. \quad (2.25)$$

Thus, relations (2.20) and (2.21) are proved for $k_{-}(x)$. In the same way relations (2.20) and (2.21) can be proved for $k_{+}(x)$. \square

3. We use the following notation

$$J(f) = J_1(f) + J_2(f), \quad (2.26)$$

where

$$J_1(f) = \frac{d}{dx} \int_{-\infty}^x f'(y)k_{-}(y-x)dy, \quad f(x) \in C(a), \quad (2.27)$$

$$J_2(f) = \frac{d}{dx} \int_x^{\infty} f'(y)k_{+}(y-x)dy, \quad f(x) \in C(a). \quad (2.28)$$

Lemma 2.4 *The operator $J(f)$ defined by (2.26) can be represented in the form*

$$J(f) = \int_{-\infty}^{\infty} [f(y+x) - f(x) - y \frac{df(x)}{dx} 1_{|y| \leq 1}] \mu(dy) + \Gamma f'(x), \quad (2.29)$$

where $\Gamma = \overline{\Gamma}$ and $f(x) \in C(a)$.

Proof. From (2.27) we obtain the relation

$$J_1(f) = - \int_{x-1}^x [f'(y) - f'(x)] k'_-(y-x) dy - \int_{-a}^{x-1} f'(y) k'_-(y-x) dy. \quad (2.30)$$

By proving (2.30) we used relations (2.16), (2.20) and equality

$$\int_{x-1}^x k_-(y-x) dy = \int_{-1}^0 k_-(v) dv. \quad (2.31)$$

We introduce the notations

$$P_1(x, y) = f(y) - f(x) - (y-x)f'(x), \quad P_2(x, y) = f(y) - f(x). \quad (2.32)$$

Using notations (2.32) we represent (2.30) in the form

$$J_1(f) = - \int_{-1}^0 \frac{\partial}{\partial y} P_1(x, y+x) \mu_-(y) dy - \int_{-a-x}^{-1} \frac{\partial}{\partial y} P_2(x, y+x) \mu_-(y) dy, \quad (2.33)$$

Integrating by parts the integrals of (2.33) we deduce that

$$\begin{aligned} J_1(f) = & f'(x) \gamma_1 + \int_{-1}^0 P_1(x, y+x) d\mu_-(y) + \int_{-a-x}^{-1} P_2(x, y+x) d\mu_-(y) \\ & + P_2(x, -a) \mu_-(-a), \end{aligned} \quad (2.34)$$

where $\gamma_1 = k'_-(-1)$. It follows from (1.3) that the integrals in (2.34) are absolutely convergent. Passing to the limit in (2.34), when $a \rightarrow +\infty$, and taking into account (2.30), (2.31) we have

$$J_1(f) = \int_{-\infty}^x [f(y+x) - f(x) - y \frac{df(x)}{dx} 1_{|y| \leq 1}] d\mu_-(y) + \gamma_1 f'(x). \quad (2.35)$$

In the same way it can be proved that

$$J_2(f) = \int_x^{\infty} [f(y+x) - f(x) - y \frac{df(x)}{dx} 1_{|y| \leq 1}] d\mu_+(y) + \gamma_2 f'(x), \quad (2.36)$$

where $\gamma_2 = k'_+(1)$. The relation (2.29) follows directly from (2.35) and (2.36). Here $\Gamma = \gamma_1 + \gamma_2$. The lemma is proved. \square

Remark 2.1 *The operator $L_0 f = \frac{d}{dx} f$ can be represented in form (2.1), (2.2), where*

$$S_0 f = \int_{-\infty}^{\infty} p_0(x-y) f(y) dy, \quad (2.37)$$

$$p_0(x) = \frac{1}{2} \text{sign}(x). \quad (2.38)$$

From Lemmas 2.4, and Remark 2.1 we deduce the following assertion.

Theorem 2.5 *The infinitesimal generator L has a convolution type form (2.1), (2.2).*

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